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Perturbations of Stable and Chaotic Difference Equations

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1. INTRODUCTION

This work continues the investigation begun in [9] concerning the dynamics of multidimensional difference schemes of the form:

$$X_{k+1} = F(X_k) \tag{1.1}$$

where $X_k \in \mathbb{R}^n$ and $F: \mathbb{R}^n \to \mathbb{R}^n$. In recent years numerous difference schemes of this form have appeared in the literature as models for a variety of physical and biological phenomena. This is most notable in the field of population biology, in which equations of this type are used to model discrete population growth in a network of n interacting species. (See the works of May for example.) It is therefore of interest to develop practical means of determining the qualitative behavior of problems of the form (1.1). The particular dynamics upon which we shall focus is the existence of either stable equilibria, stable periodic solutions, or chaos.

The conditions for (local asymptotic) stability of fixed or periodic solutions of (1.1) are well known. If we let F^k represent the composition of the function F with itself k times, then X is a periodic point of period p if $F^p(X) = X$ and $F^k(X) \neq X$ for $1 \leqslant k < p$. The collection $\{F^k(X)\}_{k=1}^{p}$ is called a p-cycle of (1.1), and if $p = 1$ then X is a fixed or equilibrium point. If X is a point of period p then X is stable under (1.1) if all eigenvalues of $DF^p(X)$, the jacobian of F^p at the point X, are less than 1 in norm.

The concept of chaos, on the other hand, is a relatively new and not very well understood phenomenon. It has been suggested by Ruelle and Takens [11] that chaos is the mathematical analogue of turbulence in the flow of fluids. Although chaotic forms of behavior had previously been observed in a variety of settings, the term "chaos" was first used by Li and Yorke [7], who presented sufficient conditions for its existence in scalar problems of the form (1.1) where $F: \mathbb{R} \to \mathbb{R}$. Although their theorem has been extensively used in explaining the complex behavior exhibited by a number of such equations, it is not applicable to multidimensional schemes.

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A result which does provide conditions for chaos of multidimensional problems was previously presented in [9]. Suppose F is differentiable and Z is an unstable fixed point of (1.1) such that, for some $r > 0$, all eigenvalues of $DF(X)$ exceed 1 in norm for all $X \in B_r(Z)$, the ball of radius r around Z. We shall say that Z is a *snap-back repeller* if there exists a point $X_0 \in B_r(Z)$ with $X_0 \neq Z$, $F^M(X_0) = Z$, and $\mathrm{Det}[DF^M(X_0)] \neq 0$.

THEOREM 1.1. *If F has a snap-back repeller then* (1.1) *is chaotic.*

(See [9] for a proof and for a precise definition of choas.) It is proven in [8] that Theorem 1.1 is (roughly) a generalization of the scalar results of Li and Yorke.

Theorem 1.1 is also closely related to some results of Smale [12]. Suppose Z is a fixed point of a diffeomorphism F with some eigenvalues of $DF(Z)$ greater than 1 in norm and the remainder of them less than 1 in norm. In this case there exist stable and unstable manifolds of F at Z. In general it is possible for these manifolds to intersect transversally at some point X_0 (other than Z). The trajectory $\{X_k\}_{k=-\infty}^{+\infty}$ under (1.1) has the properties $X_k \to Z$ and $X_{-k} \to Z$ as $k \to \infty$, and is called a *transversal homoclinic orbit*. Smale has proven the following.

THEOREM 1.2. *If F has a transversal homoclinic orbit then there exists a Cantor set $\Lambda \subset \mathbb{R}^n$ in which F^M is topologically equivalent to a shift automorphism for some positive integer M.*

The existence of such a shift automorphism implies that within Λ there exists a dense collection of periodic points (as well as other chaotic behavior). The relationship between snap-back repellers and transversal homoclinic orbits is apparent. The definition of a snap-back repeller Z implies the existence of a solution $\{X_k\}_{k=-\infty}^{+\infty}$ of (1.1) satisfying $X_k = Z$ for $k \geqslant M$, and $X_k \to Z$ as $k \to -\infty$. Such a solution is analogous to a homoclinic orbit. Also, the conditions $X_0 \in B_r(Z)$ and $\mathrm{Det}[DF^M(X_0)] \neq 0$ imply that $\mathrm{Det}[DF(X_k)] \neq 0$, and hence the mapping F is locally $1 - 1$ at each X_k for $-\infty < k < +\infty$. This is analogous to transversality of a homoclinic orbit. In fact, snap-back repellers may be viewed as a special case of a fixed point with a transversal homoclinic orbit if we generalize the latter to exist for functions which are not necessarily $1 - 1$. In this case the unstable manifold $\mathbb{R}^n$ transversally intersects the zero-dimensional stable manifold Z, producing the orbit $\{X_k\}_{k=-\infty}^{+\infty}$ where $X_M = Z$ and $X_k \to Z$ as $k \to -\infty$.

Although either type of homoclinic behavior implies some form of chaos, there are some important practical differences between transversal homoclinic orbits and snap-back repellers. In order to compute the former, extremely careful numerical calculations must be performed to first find the stable and unstable manifolds, and then show transversal intersection. This must usually be done visually with the aid of computer graphics. Snap-back repellers, on the other hand, are relatively easy to compute, often requiring only finite iteration processes. (See [9].)

It would therefore be very convenient if chaotic behavior could be proven by exhibiting snap-back repellers rather than transversal homoclinic orbits. This is the primary purpose of this work. We shall show that, in certain circumstances, the existence of a snap-back repeller of a particular equation implies the existence of a transversal homoclinic orbit of a higher dimensional problem. This will be true when the problem of higher dimension is a small perturbation of the lower dimensional system. In effect this provides a practical means of determining the existence of a transversal homoclinic orbit, and therefore chaos, in certain problem of the form (1.1), namely, by reducing the problem to one of lower dimension having a snap-back repeller and then applying perturbation methods. In particular we shall study certain two-dimensional problems by perturbing scalar equations.

Moreover, there is also considerable computational difficulty in determining the existence and stability of periodic solutions of multidimensional equations of the form (1.1), though the conditions for these are well known. We shall therefore also show that the existence of such solutions can sometimes be proven by again applying perturbation arguments to a lower dimensional problem.

These results are presented in the following section, although several non-trivial proofs are left for the Appendix. Section 3 contains applications of these ideas to several multidimensional difference schemes which have appeared in the literature. Among these are the Leslie model of Guckenheimer, Oster and Ipaktchi [3], the two-dimensional transformation of Hénon [5], and two competition models previously discussed by Hassell and Comins [4].

2. Principal Results

As previously described we shall attempt to determine the dynamics of certain more complex problems by reducing dimensions in some manner. Let us first consider a mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ of the form $F(x, y) = (f(x), x)$ where $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Suppose f has a stable point x_0 of period p, i.e., $f^p(x_0) = x_0$ and $| Df^p(x_0)| < 1$. It can be easily shown that the point (x_0, y_0) where $y_0 = f^{p-1}(x_0)$ is a point of period p for F. In addition the eigenvalues of the jacobian:

$$DF^p(x_0, y_0) = \begin{bmatrix} Df^p(x_0) & 0 \\ Df^{p-1}(x_0) & 0 \end{bmatrix}$$

satisfy $|\lambda_1| = |Df^p(x_0)| < 1$ and $|\lambda_2| = 0$. Although $DF^p(x_0, y_0)$ has non-trivial kernal, we nevertheless have proven the following.

LEMMA 2.1. *If f has a stable point x_0 of period p, then $F(x, y) = (f(x), x)$ has a stable point (x_0, y_0) of period p.*

We can thus determine the existence and stability of periodic points of the two-dimensional mapping F from an analysis of the scalar mapping f.

The mapping F is also interesting in that it shows more clearly the relationship between snap-back repellers and transversal homoclinic orbits, namely, the former may be viewed as a special case of the latter when embedded in a higher dimensional setting.

LEMMA 2.2. *If f has a snap-back repeller, then $F(x, y) = (f(x), x)$ has a transversal homoclinic orbit.*

A proof of this lemma appears in the Appendix. We remark that again $DF(x, y)$ has nontrivial kernal.

With the use of these results we can examine the dynamics of a class of difference equations of the form:

$$x_{k+1} = f(x_k, bx_{k-1}) \tag{2.1}$$

where b, $x_k \in \mathbb{R}$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Note that (2.1) can be equivalently written as a two-dimensional system:

$$\begin{aligned} x_{k+1} &= f(x_k, by_k) \\ y_{k+1} &= x_k . \end{aligned} \tag{2.2}$$

Also note that when $b = 0$, (2.1) reduces to the scalar problem:

$$x_{k+1} = f(x_k, 0). \tag{2.3}$$

We shall show that the dynamics of (2.1) or (2.2) when b is close to 0 are determined by those of (2.3) Suppose $f(x, 0)$ has a stable point x_0 of period p According to Lemma 2.1 the mapping defined by $F(x, y, 0) = (f(x, 0), x)$ also has a stable point (x_0, y_0) of period p. Now consider the mappings $F(x, y, b) = (f(x, by), x)$. Since $F(x, y, b)$ is a continuously differentiable perturbation of $F(x, y, 0)$, we can employ well known perturbation results to conclude that $F(x, y, b)$ also has a stable point of period p for small values of b. (See for example Theorem 5.1 of Hirsch, Pugh and Shub [6].) We have thus proven the following.

THEOREM 2.1. *If (2.3) has a stable point x_0 of period p, then there exists $\epsilon > 0$ such that (2.2) has a stable point $(x(b), y(b))$ of period p for all $|b| < \epsilon$. In this case $(x(b), y(b))$ is a uniquely defined, continuous function of b with $x(0) = x_0$.*

In a similar manner we can determine the existence of chaos of (2.2) by investigating the same in (2.3). If $f(x, 0)$ has a snap-back repeller, then by Lemma 2.2 $F(x, y, 0)$ has a transversal homoclinic orbit. Although $DF(x, y, 0)$ has nontrvial kernal, the proof of Lemma 2.2 shows that under iteration of $F(x, y, 0)$ a closed segment of the unstable manifold U_0 (not containing the fixed point of

$F(x, y, 0))$ intersects the stable manifold S_0 transversally. Thus the same must be true of U_0 and S_0 under iteration of $F(x, y, b)$, for all $|b| < \delta$ for some $\delta > 0$. (See for example Theorem 18.2 of Abraham and Robbin [1], concerning the openness of transversal intersection of submanifolds.) But the manifolds of $F(x, y, b)$, S_b and U_b, must be "close" to S_0 and U_0 respectively for sufficiently small b. (Again see Hirsch, Pugh and Shub [6].) Therefore a closed segment of U_b (not containing the fixed point of $F(x, y, b)$) must transversally intersect S_b under iteration of $F(x, y, b)$ for all $|b| < \epsilon < \delta$ for some $\epsilon > 0$. This proves that $F(x, y, b)$ also has a transversal homoclinic orbit, and hence (2.2) behaves chaotically. We state these results formally.

THEOREM 2.2. *If* (2.3) *has a snap-back repeller, then* (2.2) *has a transversal homoclinic orbit for all* $|b| < \epsilon$ *for some* $\epsilon > 0$.

Remark 2.1. We can extend the previous analysis by viewing the more general problem:

$$x_{k+1} = f(x_k, b_1 x_{k-1}, \dots, b_m x_{k-m}) \tag{2.4}$$

as a perturbation of the scalar equation:

$$x_{k+1} = f(x_k, 0, \dots, 0). \tag{2.5}$$

If we let b_i, $x_k \in \mathbb{R}$ and $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ be continuously differentiable we can prove the following in a manner similar to the previous results.

(1) If (2.5) has a stable point of period p, then (2.4) has a stable point of period p for all $|b_i| < \epsilon$ for some $\epsilon > 0$.

(2) If (2.5) has a snap-back repeller, then (2.4) (when written as a system) has a transversal homoclinic orbit for all $|b_i| < \epsilon$ for some $\epsilon > 0$.

The previous analysis shows that the dynamics of a certain class of two-dimensional mappings can be determined by reducing the system to a scalar equation. We shall now consider another type of two-dimensional mapping which can be reduced to two separate scalar equations. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable, and consider the mapping $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $G(x, y) = (f(x), g(y))$. The following can be easily verified.

LEMMA 2.3. *If f has a stable point x_0 of period p and g has a stable point y_0 of period q, then $G(x, y) = (f(x), g(y))$ has a stable point (x_0, y_0) of period $LCM(p, q)$.*

(Here $LCM(p, q)$ represents the least common multiple of p and q.)

Similarly we can determine the existence of chaotic behavior of G by investigating f and g separately. The following is proven in the Appendix.

LEMMA 2.4. (i) *If one of the mappings f or g has a snap-back repeller and the other has an unstable fixed point, then $G(x, y) = (f(x), g(y))$ has a snap-back repeller.*

(ii) *If one of the mappings f or g has a snap-back repeller and the other has a stable fixed point, then $G(x, y) = (f(x), g(y))$ has a transversal homoclinic orbit.*

We are now in a position to investigate the dynamics of systems of the form:

$$\begin{aligned} x_{k+1} &= f(x_k, by_k) \\ y_{k+1} &= g(cx_k, y_k) \end{aligned} \tag{2.6}$$

where $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ are differentiable, and $b, c \in \mathbb{R}$ are close to 0. When $b = c = 0$ (2.6) reduces to the uncoupled system:

$$x_{k+1} = f(x_k, 0) \tag{2.7a}$$

$$y_{k+1} = g(0, y_k). \tag{2.7b}$$

Just as $F(x, y, b) = (f(x, by), x)$ reduced to $F(x, y, 0)$, the mapping $G(x, y, b, c) = (f(x, by), g(cx, y))$ can be reduced to $G(x, y, 0, 0)$ when b and c are small. Since $G(x, y, b, c)$ is a continuously differentiable perturbation of $G(x, y, 0, 0)$, which is of the type treated in Lemmas 3.3 and 3.4, (again using [1] and [6]) we can easily prove the following.

THEOREM 2.3. *If* (2.7a) *has a stable point x_0 of period p and* (2.7b) *has a stable point y_0 of period q, then there exists $\epsilon > 0$ such that* (2.6) *has a stable point $(x(b, c), y(b, c))$ of period $LCM(p, q)$ for all $|b|, |c| < \epsilon$. In this case $x(b, c)$ and $y(b, c)$ are uniquely defined, continuous functions of b and c with* $(x(0, 0), y(0, 0)) = (x_0, y_0)$.

THEOREM 2.4. (i) *If one of the problems* (2.7a) *or* (2.7b) *has a snap-back repeller and the other has an unstable fixed point, then* (2.6) *has a snap-back repeller for all $|b|, |c| < \epsilon$ for some $\epsilon > 0$.*

(ii) *If one of the problems* (2.7a) *or* (2.7b) *has a snap-back repeller and the other has a stable fixed point, then* (2.6) *has a transversal homoclinic orbit for all $|b|, |c| < \epsilon$ for some $\epsilon > 0$.*

Remark 2.2. The previous results describe the dynamics of two classes of problems when certain parameters are close to 0. Note however that no estimate for ϵ is provided in these theorems. To some extent this is a drawback of these arguments. On the other hand, what is of interest quite often are the dynamical possibilities of a particular mathematical model as parameters are varied. This is primarily true since accurate estimates of parameters in many models, especially biological models, are very seldom obtainable. Hence, although we provide no estimate for ϵ, the theorems of this section aid in determining the range of dynamical features of certain multidimensional models.

3. Applications

APPLICATION 3.1. In this section we shall investigate several two-dimensional schemes which have appeared elsewhere in the literature. First consider the systems:

$$y_{k+1} = (ax_k + by_k)(1 - ax_k - by_k) \tag{3.1a}$$

$$y_{k+1} = x_k \tag{3.1b}$$

and

$$x_{k+1} = (ax_k + by_k)\exp(-ax_k - by_k) \tag{3.2a}$$

$$y_{k+1} = x_k \tag{3.2b}$$

where $a, b > 0$. The first is a two-dimensional generalization of the logistic equation studied by May [10]. A lengthy discussion of (3.1) appears in [9]. Equation (3.2) is a modification of the Leslie model studied by Guckenheimer, Oster and Ipaktchi [3].

When $b = 0$ (3.1a) and (3.2a) reduce to scalar equations, the dynamics of which have been extensively investigated. (See May [10] for example.] As the parameter a is increased (with $b = 0$) each equation displays similar qualitative features, including stable fixed points, stable 2^n-cycles for $n \geqslant 1$, and chaos. In particular, when $b = 0$ (3.1a) has a stable fixed point for $0 < a < 3$ and a stable 2-cycle for $3 < a < 3.449$, while (3.2a) has a stable fixed point for $0 < a < 7.389$ and a stable 2-cycle for $7.389 < a < 12.503$.

Using the results of the previous section we can conclude the existence of certain stable periodic solutions of (3.1) and (3.2) by reducing them to the respective problems with $b = 0$. Applying Theorem 2.1 shows that if $0 < a < 3$ then (3.1) also has a stable fixed point for all $|b| < \epsilon$ for some $\epsilon = \epsilon(a) > 0$. Also, if $3 < a < 3.449$ then (3.1) has a stable 2-cycle for all $|b| < \delta$ for some $\delta = \delta(a) > 0$. (Note that ϵ and δ depend upon a.) This substantiates previous numerical findings presented in [9]. In this previous work (3.1) was investigated for parameter values in the region $R = \{(a, b)\colon a, b > 0 \text{ and } a + b \leqslant 4\}$. Figure 1 depicts the type of behavior observed for small values of b and relatively large values of a. The existence of these stable periodic cycles has now been proven analytically.

Similar conclusions can be made for (3.2). That is, (3.2) must have a stable fixed point if $0 < a < 7.389$, or a stable 2-cycle if $7.389 < a < 12.503$, for all $|b| < \epsilon$ for some $\epsilon = \epsilon(a) > 0$.

As mentioned above, when $b = 0$ numerical studies suggest that (3.1a) and (3.2a) behave chaotically for large values of a ($a > 3.570$ for (3.1a) and

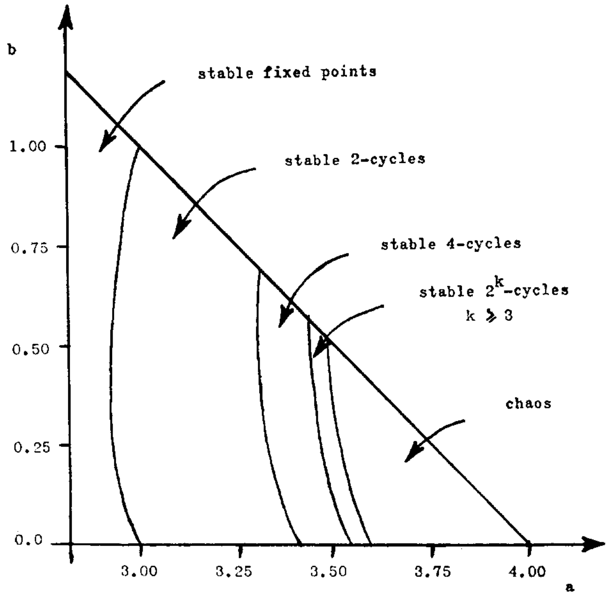

FIGURE 1

$a > 14.765$ for (3.2a) according to May). It was demonstrated in [9] that when $b = 0$ the fixed point $x_k = (a - 1)/a$ is a snap-back repeller of (3.1a) for $a > 3.680$, and the fixed point $x_k = \ln(a)$ is a snap-back repeller of (3.2a) for $a > 16.999$. Thus Theorem 2.2 implies that (3.1) and (3.2) have transversal homoclinic orbits, and hence chaos, for any value of a in the appropriate aforementioned interval, and for all $|b| < \epsilon$ for some $\epsilon = \epsilon(a) > 0$. These conclusions substantiate previous numerical observations of (3.1), illustrated in Figure 1. It was also shown in [9] that other snap-back repellers exist for (3.1a) and (3.2a) with $b = 0$ and slightly smaller values of a. Thus we may also conclude the existence of transversal homoclinic orbits for (3.1) and (3.2) for these lesser values of a and some $b > 0$.

APPLICATION 3.2. Another interesting difference scheme, originally motivated by a chaotic system of differential equations, has been investigated by Hénon [5]. With a minor change of variables this problem can be written:

$$x_{k+1} = 1 - rx_k^2 + sy_k \tag{3.3a}$$

$$y_{k+1} = x_k . \tag{3.3b}$$

It has been observed numerically that, for certain values of $r > 0$ and $s > 0$, this system exhibits chaos. We shall investigate this difference scheme using the tools developed in Section 2.

If we let $s = 0$, (3.3a) can be written:

$$u_{k+1} = au_k(1 - au_k) \tag{3.4}$$

where $a = 1 + (1 + 4r)^{1/2}$ and $u_k = (r/a^2)\, x_k + 1/2a$. The qualitative features of (3.4), which are discussed in Application 3.1 (equation (3.1a) with $b = 0$), must be the same for (3.3a) with $s = 0$. Consequently, for $s = 0$ we should expect stable 2^n-cycles for $n \geqslant 0$ or chaos of (3.3a) as r is varied. In particular, there must be a stable fixed point of (3.3a) with $s = 0$ for $0 < r < 0.75$ (corresponding to $0 < a < 3$) and a stable 2-cycle for $0.75 < r < 1.250$ (corresponding to $3 < a < 3.449$). Again using Theorem 2.1 we therefore have a stable fixed point or 2-cycle of (3.3) for each value of r in the appropriate interval and $|s| < \epsilon$ for some $\epsilon = \epsilon(r) > 0$.

Similarly we can prove the existence of chaos of (3.3) for certain parameters. Since we know (3.4) has a snap-back repeller for $a > 3.680$, (3.3a) with $s = 0$ has the same for $r > 1.546$. Thus Theorem 2.2 implies that for each such r value (3.3) has a transversal homoclinic orbit for all $|s| < \epsilon$ for some $\epsilon = \epsilon(r) > 0$. This substantiates the findings of Curry [2] whose numerical studies of (3.3) show the existence of a homoclinic orbit for certain large values of r and small values of s.

APPLICATION 3.3. Finally, let us consider two competition models which were posed by Hassell and Comins [4]:

$$\begin{aligned} x_{k+1} &= x_k \exp[r - a(x_k + \alpha y_k)] \\ y_{k+1} &= y_k \exp[s - c(y_k + \beta x_k)] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} x_{k+1} &= \lambda x_k[1 + a(x_k + \alpha y_k)]^{-b} \\ y_{k+1} &= \mu y_k[1 + c(y_k + \beta x_k)]^{-d} \end{aligned} \tag{3.6}$$

where all parameters are non-negative. Numerical investigations described in [4] indicate that each model exhibits stable fixed or periodic points, or chaos depending upon parameters. We can verify this behavior by using the results of the previous section.

It is easily seen that letting $\alpha = \beta = 0$ transforms these models into the uncoupled systems:

$$x_{k+1} = x_k \exp[r - ax_k] \tag{3.7a}$$

$$y_{k+1} = y_k \exp[s - cy_k] \tag{3.7b}$$

and

$$x_{k+1} = \lambda x_k[1 + ax_k]^{-b} \tag{3.8a}$$

$$y_{k+1} = \mu y_k[1 + cy_k]^{-d} \tag{3.8b}$$

respectively. Note that the equation for y_k in each system is the same as that for the corresponding x_k for different parameter values. The dynamics of (3.7a) (and therefore of (3.7b)) are described in Application 3.1 (equation (3.2a) with $b = 0$ and a simple change of variables) We thus know that for some fixed parameter values (3.7a) has a stable 2^n-cycle, and (3.7b) has a stable 2^m-cycle where $n \geqslant m \geqslant 0$. Hence, according to Theorem 2.3, (3.5) has a stable point of period $LCM(2^n, 2^m) = 2^n$ for these same parameters and all $|\alpha|, |\beta| < \epsilon$ for some $\epsilon > 0$.

For other fixed parameter values we know that (3.7a) has a snap-back repeller, and (3.7b) has an unstable fixed point. So, by Theorem 2.4 (3.5) also has a snap-back repeller for these same parameters and all $|\alpha|, |\beta| < \epsilon$ for some $\epsilon > 0$. Similarly, for those parameters for which (3.7a) has a snap-back repeller while (3.7b) has a stable fixed point, (3.5) has a transversal homoclinic orbit for all $|\alpha|, |\beta| < \delta$ for some $\delta > 0$. (Note that in this discussion ϵ and δ depend upon r, a, s and c.)

It can be proven that the dynamics of (3.8a) (and therefore of (3.8b)) are identical to those of (3.7a), that is, there are regions in the parameters space of stable fixed points, stable 2^n-cycles, or chaos (snap-back repellers). Consequently the dynamical features of (3.6), when α and β are small, are similar to those we have seen for (3.5). This substantiates the findings of Hassell and Comins, whose numerical investigations indicate the existence of either stable periodic solutions or chaos.

Appendix

Proof of Lemma 2.2. Suppose f has a snap-back repeller. Then there exists a point z with $z = f(z)$ and $|Df(z)| > 1$, and a collection of points $\{z_k\}_{k=-\infty}^{+\infty}$ (not all $z_k = z$) with $z_k = z$ for all $k \geqslant M$, $z_{k+1} = f(z_k)$ and $Df(z_k) \neq 0$. In [9] it was shown that there must also exist a sequence of closed intervals $\{I_k\}_{k=-\infty}^{M}$ with each z_k an element but not an endpoint of I_k, $I_k \to z$ as $k \to -\infty$, $I_{k+1} = f(I_k)$, f is $1-1$ in each I_k, and $I_i \cap I_j = \varnothing$ for $i, j < M$ with $i \neq j$. Consider the set $C = \{(x, y): x = f(y) \text{ for } y \in \mathbb{R}\}$. ($C$ is precisely the graph of $x = f(y)$.) C is invariant under F, since, for any $(x, y) \in C$, $F(x, y) = (f(x), x) \in C$. Note that the point $(z, z) \in C$ is a fixed point of F. Also, the eigenvalues λ of $DF(z, z)$ satisfy: $(Df(z) - \lambda)(-\lambda) = 0$. Thus $\lambda_1 = 0$ corresponds to the stable manifold S and λ_2, satisfying $|\lambda_2| = |Df(z)| > 1$, corresponds to the unstable manifold U. In fact these manifolds can be computed exactly.

To compute S. Since S is composed of points which approach (z, z) under iteration of F, this set is precisely $S = \{(x, y); x = z\}$. In fact, all points of S are mapped onto (z, z).

To compute U. We shall show that C is locally the manifold U. First, since

$|Df(z)| > 0$, given y' near z there exists y with $f(y) = y'$. If we let $x = f(y) = y'$ and $x' = f(y')$, we have $F(x, y) = F(y', y) = (f(y'), y') = (x', y')$. Thus for any $(x', y') = (f(y'), y') \in C$ near (z, z) there exists $(x, y) = (f(y), y) \in C$ near (z, z) with $F(x, y) = (x', y')$. In addition, since $|Df(z)| > 1$:

$$\left| \frac{z - y'}{z - y} \right| = \left| \frac{f(z) - f(y)}{z - y} \right| > 1$$

and hence $|z - y'| > |z - y|$. Also, since $x = y'$ and $x' = f(y')$:

$$\left| \frac{z - x'}{z - x} \right| = \left| \frac{f(z) - f(x)}{z - x} \right| > 1.$$

Therefore, $|z - x'| > |z - x|$, and from above $|z - y'| > |z - y|$. It is easily seen therefore that if $(x', y') \in C$ is close to $(z, z) \in C$, then there exists (x, y) even closer to (z, z) with $F(x, y) = (x', y')$. Consequently, in C the "negative limit set" of any point close to (z, z) is the point (z, z), implying that C is locally U.

Hence $S = \{(x, y): x = z\}$ and locally $U = \{(x, y) \in C: |y - z| < \epsilon$ for some $\epsilon > 0\}$. Note that S and U intersect transversally at (z, z) since along S $(dx/dy) = 0$, and along U at (z, z) $|dx/dy| = |Df(z)| > 1$.

Now consider the collection of points $\{(z_{k+1}, z_k)\}_{k=-\infty}^{+\infty}$ where each z_k is defined above. Since $z_{k+1} = f(z_k)$ we must have $(z_{k+1}, z_k) \in C$ for all k, and since $z_k \to z$ as $k \to -\infty$, we also have $(z_{k+1}, z_k) \to (z, z)$. In addition, $z_k = z$

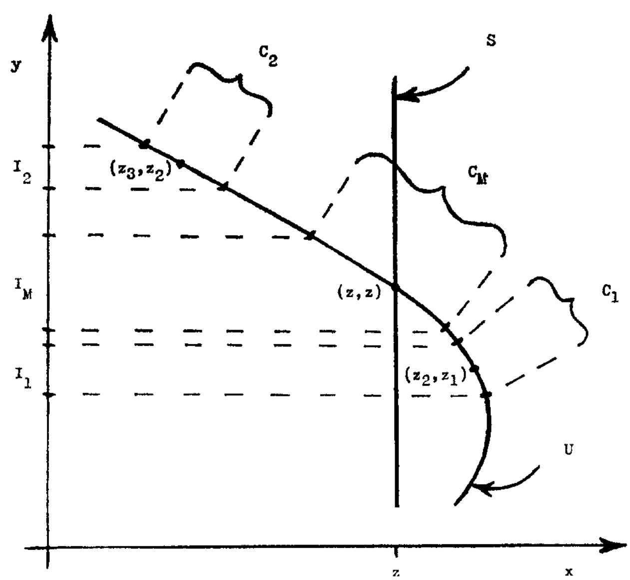

FIGURE 2

for $k \geqslant M$ implies $(z_{k+1}, z_k) = (z, z)$ for $k \geqslant M$. It is evident therefore that the sequence $\{(z_{k+1}, z_k)\}_{k=-\infty}^{+\infty}$ is a homoclinic orbit of F.

To show transversality, consider the collection $\{C_k\}_{k=-\infty}^{M}$ where $C_k = \{(x, y) \in C: y \in I_k\}$. Each C_k is that segment of C whose projection into the y-axis is I_k. Note that $C_k \to (z, z)$ as $k \to -\infty$ since $I_k \to z$ as $k \to -\infty$. Assume therefore that C_1 is sufficiently close to (z, z) so that C can be identified with U in C_1. (Relabelling the subscripts of the sequence $\{I_k\}_{k=-\infty}^{M}$ can accomplish this.) It can be easily verified that since f maps I_k onto I_{k+1} in a $1-1$ manner, F maps C_k onto C_{k+1} in a $1-1$ manner. This implies that F^{M-1} maps C_1 onto C_M in a $1-1$ manner. It can also be shown that (z_{k+1}, z_k) is an element but not an endpoint of C_k. These facts are most readily seen in Figure 2. Hence the segment C_1 of U containing the homoclinic point (z_2, z_1) is mapped in a $1-1$ manner onto C_M which intersects S transversally. This shows that $\{(z_{k+1}, z_k)\}_{k=-\infty}^{+\infty}$ is a transversal homoclinic orbit of F.

Proof of Lemma 2.4(i). Suppose x_0 is a snap-back repeller of f and y_0 is an unstable fixed point of g. So, (x_0, y_0) is a fixed point of $G(x, y) = (f(x), g(y))$. We shall show that (x_0, y_0) is a snap-back repeller of G. First, the eigenvalues of $DG(x_0, y_0)$ satisfy $(Df(x_0) - \lambda)(Dg(y_0) - \lambda) = 0$. Here, because y_0 is unstable under g, $|\lambda_2| = |Dg(y_0)| > 1$. Also, since x_0 is a snap-back repeller of f, x_0 by definition must be unstable, and so $|\lambda_1| = |Df(x_0)| > 1$. Hence, (x_0, y_0) is unstable under G.

Now since x_0 is a snap-back repeller of f, there exists a sequence $\{z_k\}_{k=-\infty}^{M}$ with $z_k \to x_0$ as $k \to -\infty$, $f(z_k) = z_{k+1}$, $z_M = x_0$ and $Df(z_k) \neq 0$. Consider the sequence $\{(z_k, y_0)\}_{k=-\infty}^{M}$. Note that $(z_k, y_0) \to (x_0, y_0)$ as $k \to -\infty$, $G(z_k, y_0) = (z_{k+1}, y_0)$, and $(z_M, y_0) = (x_0, y_0)$. If we can show that $\operatorname{Det}[DG(z_k, y_0)] \neq 0$ this will imply that (x_0, y_0) is a snap-back repeller of G. The eigenvalues λ of $DG(z_k, y_0)$ satisfy: $(Df(z_k) - \lambda)(Dg(y_0) - \lambda) = 0$. But from above $|Dg(y_0)| > 1$ and $Df(z_k) \neq 0$, thus proving that neither eigenvalue is 0. Hence $\operatorname{Det}[DC(z_k, y_0)] \neq 0$ and (x_0, y_0) is a snap-back repeller.

Proof of Lemma 2.4(ii). Again let x_0 be a snap-back repeller of f, but now let y_0 be a stable fixed point of g. So we have $|Df(x_0)| > 1$ and $|Dg(y_0)| < 1$. Also, as above, there is a sequence $\{z_k\}_{k=-\infty}^{+\infty}$ satisfying: $z_k \to x_0$ as $k \to -\infty$, $z_k = x_0$ for $k \geqslant M$, and $Df(z_k) \neq 0$. In addition, as in the proof of Lemma 2.2 we must have a sequence of closed intervals $\{I_k\}_{k=-\infty}^{M}$ with each z_k an element but not an endpoint of I_k, $I_k \to x_0$ as $k \to -\infty$, $I_{k+1} = f(I_k)$, f is $1-1$ in I_k, and $I_i \cap I_j = \varnothing$ for $i, j < M$ with $i \neq j$.

Note that (x_0, y_0) is a fixed point of $G(x, y) = (f(x), g(y))$, and that the eigenvalues λ of $DG(x_0, y_0)$ satisfy: $(Df(x_0) - \lambda)$. $(Dg(y_0) - \lambda) = 0$. Hence $|\lambda_1| = |Df(x_0)| > 1$ and $|\lambda_2| = |Dg(y_0)| < 1$. The corresponding stable and unstable manifolds are easily computed: locally, $S = \{(x, y): x = x_0\}$ and $U = \{(x, y): y = y_0\}$.

The remainder of the proof is now similar to that of Lemma 2.2. It is apparent that $\{(z_k, y_0)\}_{k=-\infty}^{+\infty}$ is a homoclinic orbit of G. To show transversality let $C_k = \{(x, y): y = y_0 \text{ and } x \in I_k\}$, and consider $\{C_k\}_{k=-\infty}^{M}$. Since $C_k \to (x_0, y_0)$ as $k \to -\infty$, assume that C_1 is sufficiently close to (x_0, y_0) so that C_1 can be identified with a segment of U. It can be easily verified that G maps C_k onto C_{k+1} in a $1-1$ manner, and that (z_k, y_0) is an element but not an endpoint of C_k. Thus the mapping G^{M-1} from C_1 onto C_M is $1-1$. See Figure 3. Also, C_M intersects S transversally, proving that $\{(z_k, y_0)\}_{k=-\infty}^{+\infty}$ is a transversal homoclinic orbit.

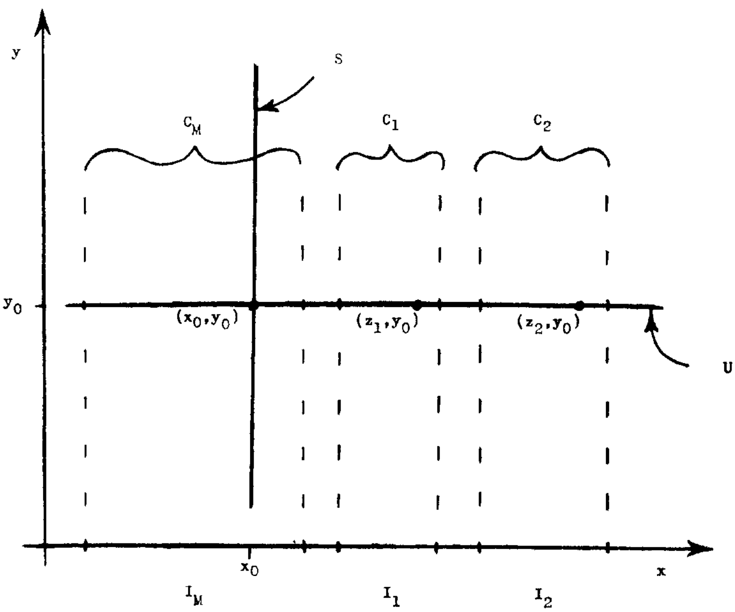

FIGURE 3

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